Conference on PDEs

University of Sussex

Geometric Analysis, Calculus of Variations,
& Harmonic Analysis

15th-17th September 2014

Abstract of the Talks

Invited Talks

Jonathan Bennett

Generating monotone quantities for the Euclidean heat equation

The identification of functionals which vary monotonically as their inputs flow according to a given evolution equation is generally considered to be more of an art than a science. Such monotonicity results have many consequences and, in particular, allow for a deeper understanding of a variety of important inequalities in analysis (via the so-called "semigroup method" or "semigroup interpolation"). The purpose of this talk is to describe a simple framework within which a rich variety of monotone quantities for the heat equation on $\mathbb{R}^n$ may be "generated".

Stefano Bianchini

The decomposition of optimal transportation problems with convex cost

Given a positive l.s.c. convex function $\mathsf{c} : \mathbb{R}^d \to \mathbb{R}^d$ and an optimal transference plane $\underline{\pi}$ for the transportation problem

$$\int \mathsf{c}(x' - x)\pi(dxdx'),$$

we show how the results on the existence of a *Sudakov decomposition* for norm cost $\mathsf{c} = |\cdot|$ can be extended to this case. More precisely, we prove that there exists a partition of $\mathbb{R}^d$ into a family of disjoint sets $\{S^h_\mathfrak{a}\}_{h,\mathfrak{a}}$ together with the projection $\{O^h_\mathfrak{a}\}_{h,\mathfrak{a}}$ on $\mathbb{R}^d$ of proper extremal faces of c, $h = 0, \dots, d$ and $\mathfrak{a} \in \mathfrak{A}^h \subset \mathbb{R}^{d-h}$, such that

- $S^h_\mathfrak{a}$ is relatively open in its affine span, and has affine dimension h;
- $O^h_\mathfrak{a}$ has affine dimension h and is parallel to $S^h_\mathfrak{a}$;
- $\mathcal{L}^d(\mathbb{R}^d \setminus \cup_{h,\mathfrak{a}} S^h_\mathfrak{a}) = 0$, and the disintegration of $\mathcal{L}^d$, $\mathcal{L}^d = \sum_h \int \xi^h_\mathfrak{a} \eta^h(d\mathfrak{a})$, w.r.t. $S^h_\mathfrak{a}$ has conditional probabilities $\xi^h_\mathfrak{a} \ll \mathcal{H}^h \llcorner_{S^h_\mathfrak{a}}$;
- the sets $S^h_\mathfrak{a}$ are essentially cyclically connected and cannot be further decomposed.

The last point is used to prove the existence of an optimal transport map. The main idea is to recast the problem in $(t, x) \in [0, \infty] \times \mathbb{R}^d$ with an 1-homogeneous norm $\bar{\mathsf{c}}(t, x) := t\mathsf{c}(-\frac{x}{t})$ and to extend the regularity estimates of the norm cost case to this case.

Anthony Dooley
Lie Symmetry Groups of PDE's, harmonic analysis and SDE's

Sophus Lies third paper was an attempt to create a version of Galois theory for PDEs, and resulted in what is known today as the method of Lie symmetry groups: they are not so well known in Pure Mathematics as perhaps they should be. For example, the Lie symmetry group of the one dimensional heat equation is the semi-direct product $G = H_1 \rtimes SL(2, \mathbb{R})$, where H_1 is the 3-D Heisenberg group. This observation leads to the use of methods from non-commutative harmonic analysis in PDE theory. One can obtain for example the Gaussian fundamental solution from the constant solution, and indeed there is an irreducible representation of G on the space of solutions. Recently, we have also extended the action of Lie symmetry groups to the realm of Stochastic Differential Equations, so that they act on random flows.

Maria Esteban
Symmetry and symmetry breaking for optimizers of functional inequalities

In this talk I will present a series of results on the symmetry properties of optimizers of functional inequalities which are invariant under a certain symmetry group. The symmetry issue is of big importance in many of the applications of those inequalities, and also in the study of many physical systems for which knowing when the symmetry is broken is of the utmost importance. The results that I will present are mainly theoretical, showing in which cases one can prove symmetry and symmetry breaking, and by which methods. But in order to understand some of the results, some numerical computations have been very useful for us and therefore I will also explain the interesting results that they suggest.

Francois Hamel
Semilinear elliptic equations in convex domains and convex ring

In this talk, I will discuss some geometrical properties of positive solutions of some semilinear elliptic equations in bounded convex domains or convex rings, with Dirichlet-type boundary conditions. A solution is called quasiconcave if its superlevel sets are convex. I will present two counterexamples, that is, two cases of semilinear elliptic equations for which the solutions are not quasiconcave. This talk is based on a joint work with N. Nadirashvili and Y. Sire.

Emmanuel Hebey
Stationary Kirchhoff systems in closed manifolds

We discuss several issues about stationary critical Kirchhoff systems in closed manifolds, such as the questions of existence and nonexistence of solutions, or the question of the compactness and stationary stability of the systems. This is joint work with Pierre-Damien Thizy.

Frédéric Hélein
Conserved quantities for hyperbolic non linear equations

Given a hyperbolic non linear equation, we address the question of finding infinitely many first integrals or conserved quantities, obtained by integrating Cauchy data along a space-like hypersurface. The answer is quite easy for linear equations. For non linear equations similar constructions are possible. They are however more complex and requires the use of series expansions. We present here a framework for such a construction, which shares some similarities with the perturbation theory in quantum fields theory.

Jan Kristensen
The Morse-Sard theorem, generalized Luzin property and level sets for Sobolev functions

Many classical results from multivariate calculus can be generalized to suitable Sobolev functions that need not even be everywhere differentiable. In this talk we discuss some new results that have been obtained in joint work with Jean Bourgain (Princeton) and Mikhail Korobkov (Novosibirsk).

Ari Laptev

Spectral inequalities for non-self-adjoint partial differential operators

We shall present some result on location of the spectrum of Dirac and Schrödinger operators with complex-valued potentials.

Yanyan Li

The Nirenberg problem and its generalizations: A unified approach

Making use of integral representations, we develop a unified approach to establish blow up profiles, compactness and existence of positive solutions of the conformally invariant equations $P_\sigma(v) = Kv^{\frac{n+2\sigma}{n-2\sigma}}$ on the standard sphere S^n for $\sigma \in (0, n/2)$, where P_σ is the conformal fractional Laplacian of order 2σ. Finding positive solutions of these equations is equivalent to seeking metrics in the conformal class of the standard metric with prescribed certain curvatures. When $\sigma = 1$, it is the prescribing scalar curvature problem of the Nirenberg problem, and when $\sigma = 2$, it is the prescribing $Q-$curvature problem. This is a joint work with Tianling Jin and Jingang Xiong.

Giuseppe Mingione

Recent progresses in nonlinear potential theory

I will present some recent results obtained in collaboration with Tuomo Kuusi (Aalto University) concerning new pointwise estimates for solutions to possibly degenerate quasilinear equations.

Andre Neves

Min-max theory in geometry

I will talk about my recent work with Fernando Marques, where we used min-max theory to solve some open problems in Geometry.

Jey Sivaloganathan
Detecting Instability in Nonlinear Elasticity

We first review some analytical and physical implications of the notion of quasiconvexity as a condition for local stability in nonlinear elasticity. Then we present a related approach to predicting when local fracture is energetically favoured in a hyperelastic material based on a novel notion of the derivative of the stored energy functional with respect to discontinuous deformations. Let $\Omega \subset \mathbb{R}^n$ be the region occupied by a hyperelastic body in its reference configuration. Consider the displacement boundary value problem in which the admissible deformations $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$ of Ω satisfy the linear boundary condition $\mathbf{u}(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \partial\Omega$ for some constant matrix $A \in M_+^{n\times n}$ ($n \times n$ matrices with positive determinant). Each deformation has an associated energy

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))dx$$

where $W : M_+^{n\times n} \to \mathbb{R}$ is the stored energy function of the material. Given $1 \leq p \leq \infty$ the stored energy function W is said to be $W^{1,p}-$quasiconvex at the matrix A provided that

$$E(\mathbf{u}) \geq E(\mathbf{u}^h)$$

for all such deformations $\mathbf{u} \in W^{1,p}(\Omega)$, where $\mathbf{u}^h \equiv A\mathbf{x}$. We restrict attention to deformations $\mathbf{u}$ which have at most one point of discontinuity at a given point $\mathbf{x}_0 \in \Omega$. In this class it is known that, under suitable assumptions on W, the unique energy minimiser is $\mathbf{u}^h$ for "small A" but any minimiser must produce a discontinuity at $\mathbf{x}_0$ for "large A". We denote by $\mathcal{S}$ the stable region (within the set of all matrices A) inside which the stored energy function is $W^{1,p}-$quasiconvex on the above class of deformations. The boundary of this region then represents a surface" in the space of strains in the sense that: given an equilibrium solution $\mathbf{u}$ which is C^1 at some $\mathbf{x}_0 \in \Omega$, if $\nabla\mathbf{u}(\mathbf{x}_0)$ lies in $\mathcal{U} := M_+^{n\times n}\backslash\bar{\mathcal{S}}$, then forming a discontinuity at $\mathbf{x}_0$ can further lower the total stored energy. We present an approach for calculating $\partial\mathcal{S}$: for each matrix A we define a variational derivative, denoted, $G(A)$ of the energy functional with respect to a hole-producing deformation. We are able to show in various cases that:

1. $\mathcal{S}$ is characterised by the condition $G(A) > 0$,
2. $G(A) < 0$ in $\mathcal{U}$,

3. ∂S is characterised by the condition $G(A) = 0$.

Hence, the bifurcation equation $G(A) = 0$ appears to characterise the fracture surface.

Alexander Sobolev

Functions of self-adjoint operators in ideals of compact operators: applications to the entanglement entropy

Let A be a self-adjoint operator, and let P be an orthogonal projection. We study properties of the difference $f(PAP) - Pf(A)P$ in arbitrary quasi-normed ideals $\mathfrak{S}$. The focus is on non-smooth functions f. A typical example is the function $f(t) = |t|^\gamma$ with $\gamma \in (0, 1)$. We apply the obtained results to study the entanglement entropy of free fermions.

Neshan Wickramasekera

Higher multiplicity in minimal submanifolds

For multiplicity 1 classes of minimal submanifolds (which include, for instance, area minimising hypersurfaces), there is a rich regularity and singularity theory going back to the work of De Goirgi, Allard and Simon. Multiplicity 1 hypothesis however is satisfied only in special cases. In this talk, I will survey parts of this classical theory and describe partial progress made in the last several years on the regularity issues and the structure of singularities in the absence of the multiplicity 1 hypothesis.

Jim Wright

Affine invariant harmonic analysis

We survey recent progress on this developing area of harmonic analysis.

Contributory Talks

Shabnam Beheshti
Controlling a family of dressed harmonic maps in general relativity

Integrability and dressing techniques have been extensively utilised in the construction and analysis of exact solutions to the Einstein vacuum and Einstein-Maxwell equations. We generalise this framework to gravitational and electrodynamic field theories which can be cast as axially symmetric harmonic maps into symmetric spaces. We shall then discuss how a control-theoretic approach can be used to establish existence–and possible non-existence–of certain singular configurations for solutions to Einstein's Equations. The main ideas in this application provide a first step in the study of more general (integrable) geometric field theories.

Filippo Cagnetti
The rigidity problem for symmetrization inequalities

Steiner symmetrisation is a very useful tool in the study of isoperimetric inequality. This is also due to the fact that the perimeter of a set is less or equal than the perimeter of its Steiner symmetral. In the same way, in the Gaussian setting, it is well known that Ehrhard symmetrisation does not increase the Gaussian perimeter. We will show characterisation results for equality cases in both Steiner and Ehrhard perimeter inequalities. We will also characterise rigidity of equality cases. By rigidity, we mean the situation when all equality cases are trivially obtained by a translation of the Steiner symmetral (or, in the Gaussian setting, by a reflection of the Ehrhard symmetral). We will achieve this through the introduction of a suitable measure-theoretic notion of connectedness, and through a fine analysis of the barycenter function for a special class of sets. These results are obtained in collaboration with Maria Colombo, Guido De Philippis, and Francesco Maggi.

Sergey Mikhailov

Scaling of localized boundary-domain integral operators for variable-coefficient PDEs

In this talk, we will give an overview of some Localised Boundary-Domain Integral Equations (LBDIEs) associated with boundary-value problems for variable-coefficient PDEs and discuss how the Localised Boundary-Domain Integral Operator (LBDIO) norms depend on a scaling parameter of the localising function. Even when the considered LBDIOs are invertible for any size of the characteristic domain of localisation (i.e., for any positive value of the scaling parameter), it appears that the norms of the operators and of their inverses, in standard Sobolev (Bessel-potential) spaces, can grow as the scaling parameter decreases. This effect may be responsible for the deterioration of convergence of the mesh-based and mesh-less numerical methods of LBDIE solution, observed for fine meshes (large number of collocation points). To refine the analysis we introduce parameter-dependent Bessel-potential spaces, where the norms of some of the considered operators behave more favourably, which facilitates an optimal choice of the LBDIEs suitable for numerical implementations.

Bianca Stroffolini

A p-caloric approximation lemma and applications to parabolic systems

We present a refinement of the method of Parabolic Lipschitz truncation due to Kinnunen and Lewis. This new approach preserves boundary data and is used to prove the p-caloric approximation. It is the generalisation to the parabolic setting of De Giorgi approximation regularity method. Previous contributions are due to Duzaar, Mingione and Steffen.