Ergodicity of the Gauss map

BSc/MMath Project

During his investigations of continued fractions in the 1800s, Gauss introduced the map $T : [0,1] \to [0,1]$,

$$T(x) = \begin{cases} \{1/x\} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $\{\cdot\}$ denotes the fractional part function. Starting from a point $x \in [0,1]$ and then successively applying the map T, Gauss observed that the iterates $T^n(x)$ exhibit the limiting behaviour

$$\lim_{n\to\infty} \lambda\left(\{x \in [0,1] : T^n(x) \in [a,b]\}\right) = \int_a^b \frac{1}{(\log 2)(1+t)}\,\mathrm{d}t \quad \text{for any interval } [a,b] \subseteq [0,1],$$

where λ is the Lebesgue measure; today a probabilist would say that $T^n(x)$ converges in distribution to a random variable with density function $\frac{1}{(\log 2)(1+t)}$ on the unit interval. Estimating the rate of convergence became known as Gauss' problem, and took over 100 years to solve.

In modern terminology, the Gauss map T is a nice example of an ergodic transformation, and plays a central role in a fruitful connection between dynamical systems and number theory. The solution of Gauss' problem led to the development of the transfer operator method, which has found many other applications since. The aim of the project is to explore topics in contemporary mathematics such as ergodicity, mixing, Perron–Frobenius operators and spectral theory using the classical example of the Gauss map.

During the project, we will:

- understand the role of the Gauss map in continued fractions,
- prove that the Gauss map is ergodic,
- understand the role of transfer operators in the solution of Gauss' problem,
- explore some applications to number theory.

Relevant modules: Analysis 1, Analysis 2, Measure Theory with Applications

References

[1] Manfred Einsiedler and Thomas Ward: *Ergodic Theory with a View Towards Number Theory.* Springer-Verlag, London, 2011.

[2] Marius Iosifescu and Cor Kraaikamp: *Metrical Theory of Continued Fractions.* Kluwer Academic Publishers, Dordrecht, 2002.